Local Dimension of Complex Networks

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Abstract

Most of the understanding of topological features in complex networks came from the direct measurements taken over nodes by means of statistical methods. This is no different from other physical complex systems, which may present a diverse range of characteristics. Dimensionality is one of the most important properties of such systems, though only recently this concept was been considered in the context of complex networks. In this paper we further develop the previously introduced concepts of dimension in complex networks by presenting a new method to characterize the dimensionality of individual nodes. The methodology consists in obtaining patterns of dimensionality at different scales of local regions for each node, which can be used to detect regions with distinct dimensional structures as well as borders. We also applied this technique to power grid networks, showing, quantitatively, that the continental European power grid is much more planar than the network covering the western states of US, which present topological dimension higher than the intrinsic embedding space dimension.

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I. INTRODUCTION

Dimension is one of the most basic concepts in Physics. Diffusion[1, 2], waves propagation[3], Brownian motion[4] and various others physical processes are highly influenced by the dimension in which those phenomena take place. Dimensionality also allows us to quantify the degrees of freedom in a system, as well to characterize the macroscopic dynamics on complex systems by means of statistical mechanics[5].

Representation of complex systems by complex networks[6–9] has proven to be very successful to describe their various features without losing their intrinsic complexity. Important physical dynamics, like diffusion and information propagation can take place in such structures. Surprisingly, not much attention has been given to characterizing the dimensionality of complex networks.

Some early attempts on describing the dimensionality of networks were successful[10–12] when considering well-known regular lattices and small graphs. Characterization of the dimensionality of complex networks was first introduced in [13], and was further developed by Gastner and Newman in [14], by proposing a new and flexible way of calculating the dimension of arbitrary networks taking into account only its topology. Recently, Daqing et al [15] introduced three novel methods to obtain the dimensionality of networks, which, yields the same results even taking into account different dynamics: diffusion, random walks and percolation. They found that the dimension values neither depends on the size nor the average degree of networks. This methodology was applied to geographical network models and real networks, including the european power grid network which was show to have dimension 2 and the world airline network dimension 3.

The dimension measurements proposed in [14] and [15], provides good insights about the global dimensional structure of networks, but fails to characterize the nodes individually. Interdependent networks, for example, may encompass networks with distinct dimensions. In this case the measurement of global dimension may not represent all nodes on the network.

In this paper we further investigate the use of dimensionality measurements to characterize two power grid networks, namely: the continental european network (*EU power grid*) and the western states power grid of the United States (*US power grid*). To characterize the dimensionality of individual nodes in a network, we propose a extension for the measurement of the dimension, which take into account the local region at different distances from a node.

We also illustrate the application of this methodology to understand how is the spreading of nodes when a network is embedded in a higher dimensional space.

II. DIMENSION OF COMPLEX NETWORKS

The dimension of a network can be understood as the minimum d_e for which the entire network can be embedded in a d_e-dimensional space without losing its topological structure. This concept was early introduced by Erdös[10], proving that for any graph G, $d_e(G) \leq 2\chi(G)$ where $\chi(G)$ is the minimum number of colors to fill all vertices in G in a way that no adjacent vertices have the same color. Because obtaining $\chi(G)$ or $d_e(G)$ is a NP-Complete[16] problem, this method can not be applied directly to obtain the dimension for large complex networks.

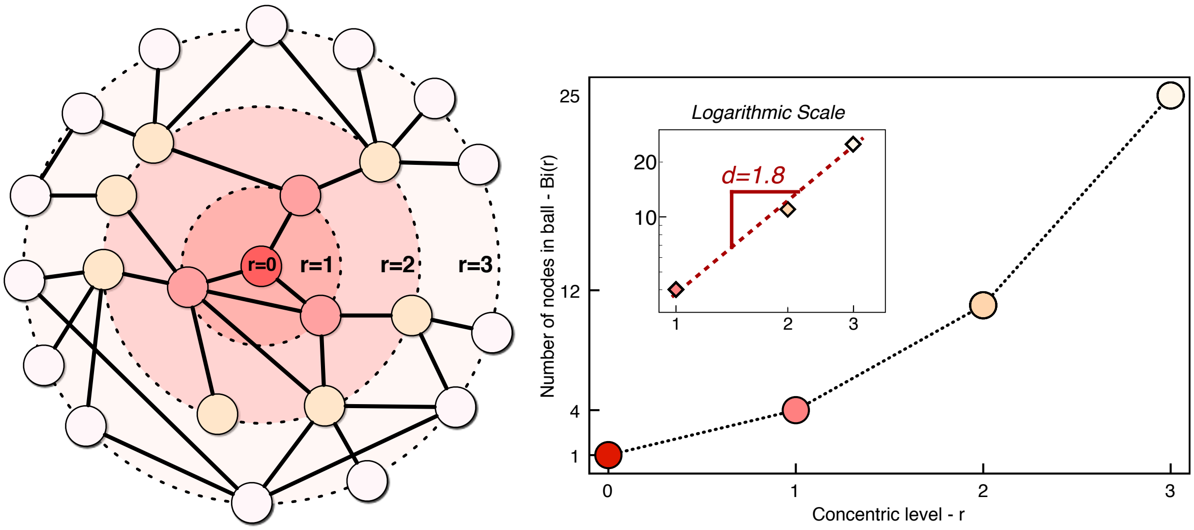
Most of the space embedded real networks that also are not small-world present length distributions that follows a power-law[17–19]. This means that for each concentric ball[20, 21], $B_i(r)$, centralized on a node i the number of nodes within obeys the equation 1 with regard to the topological distance r.

$$B_i(r) \sim r^d \tag{1}$$

The constant d is called dimension because on a infinite n-dimensional lattice, d correspond exactly to n. In fact, the dimension d characterizes the diffusion processes on networks much like in the same way as the spatial dimension characterizes the diffusion on regular spaces. For example, a diffusion process occurring on a network with $d = 2$ will present the same pattern of diffusion as if it was occurring on a 2D-plane. A estimative value of the dimension of networks can be obtained by linear regression of $B_i(r)$ on the log-log scale considering a sample of central nodes[15].

III. LOCAL DIMENSION OF NODES

While the concept of global dimension provides important characteristics about the embedding space and dynamical process on networks, much more rich information can be obtained by characterizing not only the dimension of the entire network but also the *local dimension of nodes*.



(a) (b)

FIG. 1: Example of a network represented by concentric balls, which are depicted as filled circles in (**a**), starting from a central node up to the concentric level $r = 3$. The number of nodes on each concentric ball is presented in (**b**), the dimension for this network is obtained from the angular coefficient of the double logarithmic curve shown in the inset.

Because of the heterogeneous topology structure found in most real world networks, the distribution of nodes $B_i(r)$ along the concentric distance r may vary greatly for individual nodes as starting points [21] and may not follow a strict power law. Those networks display multi-dimensional structure, therefore it is not possible to assign a unique dimension value for the entire topology, for example, in the outskirts of a city, most of the social interactions are embedded on a 2D-plane, while in downtown the social interactions may take place on large 3D-dimensional buildings, as seen in figure 2a.

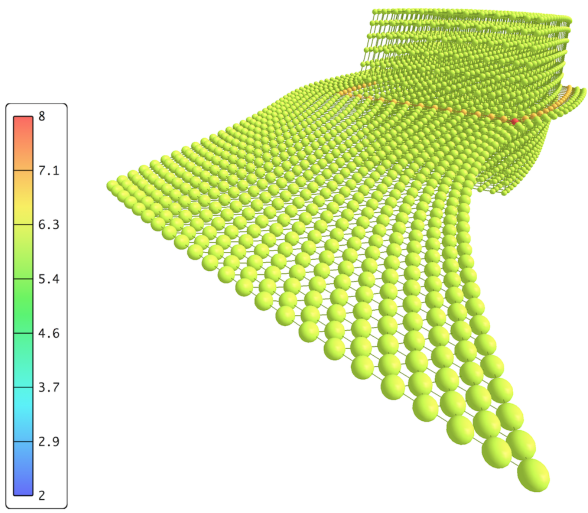
We can further develop equation 1 considering that the dimension, $D_i(r)$, may vary both for each starting node i as well for each concentric level r. The dimension coefficient can be obtained by the slope of the $B_i(r)$ curve on a double logarithmic scale, as shown:

$$B_i(r) = \alpha\, r^{D_i(r)} \tag{2}$$

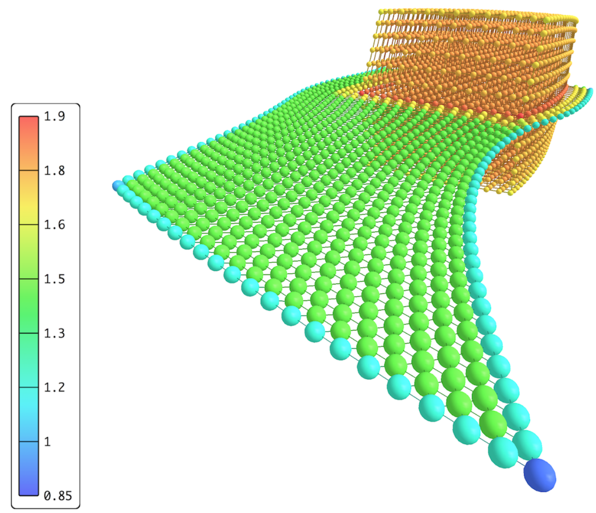
$$\log B_i(r) = D_i(r) \log r + \text{constant} \tag{3}$$

$$D_i(r) = \frac{d}{d \log r} \log B_i(r) \tag{4}$$

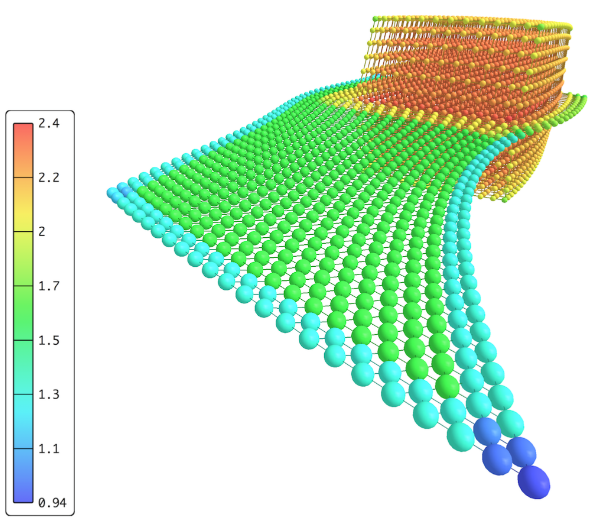
The derivative in equation 4 can be expressed in terms of r and $B_i(r)$. Because of the



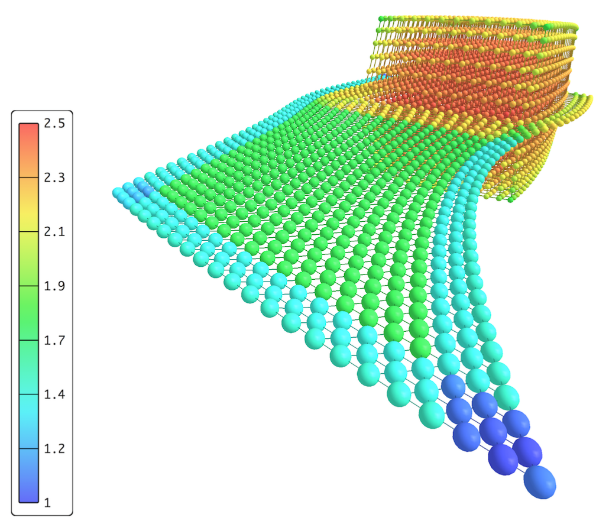
(a) Node degree.



(b) Local dimension for $r = 2$.



(c) Local dimension for $r = 3$.



(d) Local dimension for $r = 4$.

FIG. 2: Example of mixed regular networks simulating the social neighborhood in a city with two intrinsic dimensional structures: the downtown represented by the 3D region and the outskirts by a 2D surface. The network was generated in a way that node degree (**a**) is constant along all the topology. Different levels of local dimension are shown (represented by a spectral color scale in **b-d**), all of them highlight the two distinct structures on the network. Because of the discrete nature of networks, the dimension scale may be lower than the expected dimension of planar and 3D embedding spaces.

discrete nature of such measurements, the derivative can also be discretized as follows:

$$D_i(r) = \frac{r}{B_i(r)} \frac{d}{dr} B_i(r) \tag{5}$$

$$D_i(r) \simeq r \frac{n_i(r)}{B_i(r)} \tag{6}$$

where $n_i(r)$ is the number of nodes on the ring at concentric level r, i.e. the number of nodes distancing exactly r from the central node.

Even considering the approximation on equation 6, which took into account the discrete nature of complex networks, it is important to note that the equation is still valid in the case of continuous surfaces. Considering the special case of the continuous 2D plane, where $n(r) = 2\pi r$ and $B(r) = \pi r^2$ the equation yields to the expected result $D = 2$.

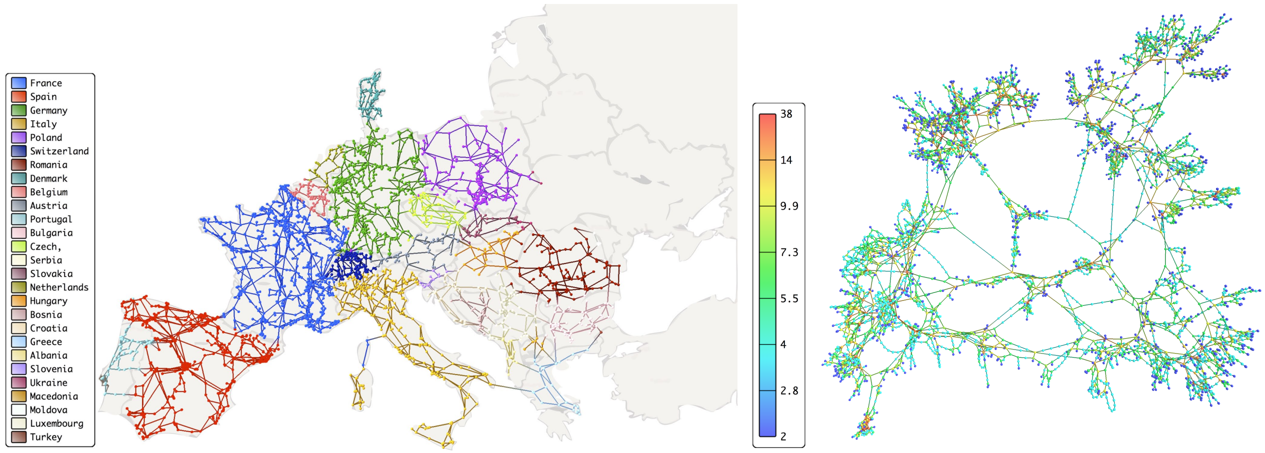
A example of generated multi-dimensional network is shown in figure 2, alongside with the calculations of local dimension $D_i(r)$ for $r = 1$, $r = 2$ and $r = 3$. Considering distances from starting nodes up to the second concentric level, i.e. $r = 2$, the measurement of local dimension, $D_i(2)$, already set apart the two distinctive dimensional regions, even considering nodes with the same degree. It is important to note that because of the finite size of the network, nodes near the frontier trends to present lower values of dimensionality.

IV. DIMENSION OF POWER GRID NETWORKS

Power grid networks are geographical structures already embedded in a 2D space, however they may feature a small number of long range connections, as well nodes density not homogeneously distributed over the plane. These complex characteristics may give rise to a considerable difference between the embedding and topological dimensions on such networks.

The EU power grid[22] network encompasses the entire continental european area considering the year 2003, covering most of the stations and power lines previously coordinated by the Union for the Coordination of Transmission of Electricity (UCTE), it is made of 2783 nodes and has $\langle k \rangle = 2.8$. The US power grid network[23] was been taken from the Pajek dataset [1], which covers the western states of United States, totalizing 4941 nodes and with $\langle k \rangle = 5.3$. Both networks projections are shown in figure 5.

[1] `http://vlado.fmf.uni-lj.si/pub/networks/data/mix/mixed.htm`



(a) EU Power Grid.

(b) US Power Grid.

FIG. 3: Topological structure of the power grid networks. Countries covered by the EU power grid network are depicted in (**a**) with colors according to the legend. The US Power Grid (**b**) is displayed with colors representing the degree of each node and projected by a force-directed method, mostly preserving the topology of the network.

Using equation 6, we obtained curves of local dimension $D_i(r)$ for every node on the power grid networks. Next, we calculated the average value of local dimension taken over the nodes, $\langle D(r) \rangle$, resulting in a curve of dimensionality pattern for each network, which are shown in figure 4. Much like the results already shown in [15]. The EU power grid network is characterized by a peak with maximum dimension value near 2 ($D_{max} = 2.14$ with our method), indicating that the network is very planar and may be embedded on a 2D surface without losing most of its topological features. On the other hand the maximum dimension obtained for the US power grid is $D_{max} = 2.7$, pointing out that some networks may present topological dimension different from their original embedded space dimension.

The curves shown in figure 4 also provide important statistical information about the dimensional behavior of diffusion processes and self avoiding random walks along different distances from a central node. Both power grids present a fast reachable peak of dimensionality, the curve for EU network follows a mostly linear slow decay, differently from the concave (and faster) decay observed for the US network. We see that for very far distances the diffusion takes place on a much more restricted set of nodes, which leads to less degrees

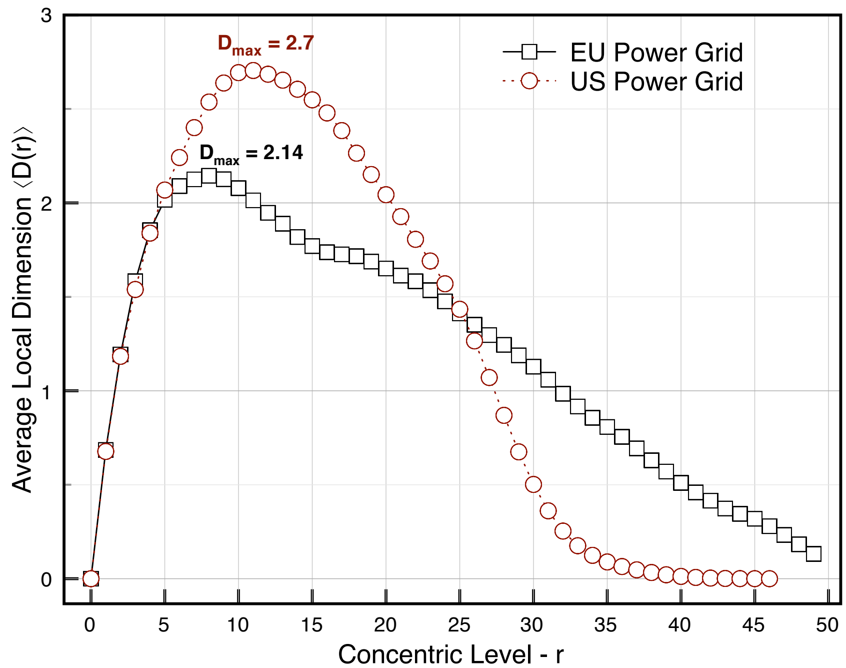


FIG. 4: Average local dimension obtained from equation 6 considering all nodes of the power grid networks. The peak of the curves characterizes the maximum dimensionality of such networks. As previously suggested the embedding space dimension may differs from the topological dimension.

of freedom and consequently lower dimension, this effect occurs much earlier for the US power grid than for the EU network.

V. DIMENSIONAL CHARACTERIZATION OF NODES

Local dimension methodology can also be applied to characterize nodes themselves, as previously exemplified in figure 2. While, real networks may not present a clear frontier between different dimensional regions, local dimensionality measurements may provide further information about the topology around nodes or regions of interest.

By embedding a network on a n-dimensional space, we may observe the correspondence relation between local topological dimension and the spreading of nodes across each degree of freedom (or axis) available on this space. A two-dimensional lattice or manifold, for example, will present no local positional spreading over a third axis when embedded on the 3D space. In general, if the dimension of the embedded space is higher than the topological

dimension we can guarantee that the network topology is well depicted on this space.

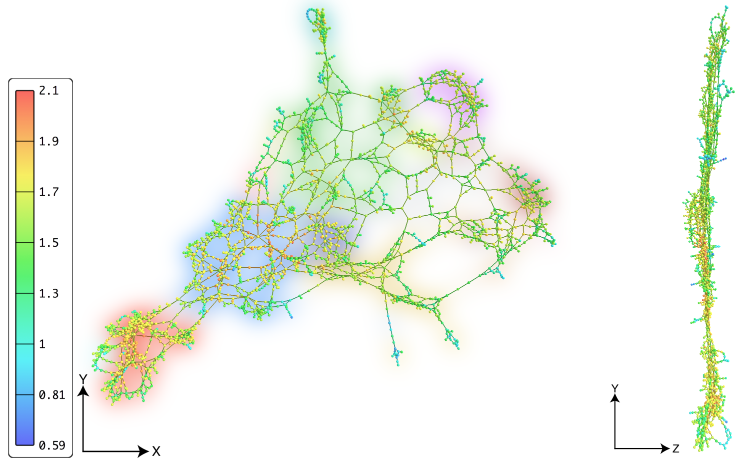
To illustrate this idea we employed the *Fruchterman-Reingold*[24] (FR) force-directed method to embed the power grid networks on the 3D space. This approach allow us not only to understand about the spreading of nodes across each axis but also to effectively visualize the local dimension distribution for the network.

Projections of the power grid networks on the 3D space are shown in figure 5, local dimension for each node considering the concentric level $r = 5$ are represented by colors. As expected, both networks seem to present two-dimensional geographical nature, with nodes spreading much more on the plane XY than on the axis Z. This effect is even more apparent for the EU network which is more planar than the US network, corroborating the previous results that the dimension of EU network is 2 while US power grid presents dimension between 2 and 3. Nodes with high dimensionality are mostly distributed on dense regions of the networks. On the EU power grid network, Spain and France are characterized by very high dimensionality, in contrast, eastern Europe present very low dimensionality. This may be the consequence of the fact that most of power plants are located in France and Spain region.

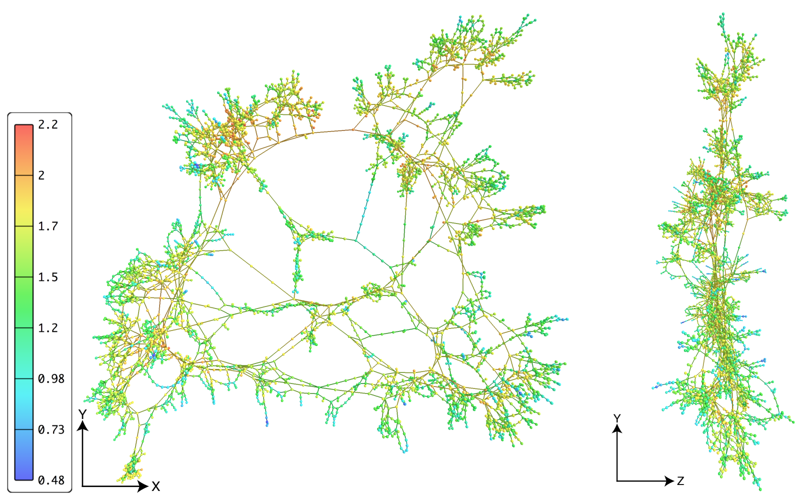
Nodes with high local dimension became centers of the spanning of nodes across all three dimensions, this behavior is more clearly observed in the US power grid because it is characterized by higher dimensionality. Another observation is that the borders of the networks, in general, present very lower dimension compared with the rest of the network, which is a reflect of what happens in finite regular structures, on a cube, for example, while the its internal topology may be characterized by dimension 3, its borders are surfaces, mostly characterized by dimension 2.

VI. CONCLUDING REMARKS

We extended the methodology to characterize the dimension of networks and introduced an experimental measurement of local dimensionality of nodes in a network, thus providing more information not only about the global overview of the distribution of dimension along networks, but also about the topology around individual nodes. This method allowed us to identify two distinct regions of dimensionality in coupled regular networks and study the heterogeneous distribution of dimensionality on power grid networks. While the EU power



(a) EU Power Grid - front view (b) side view



(c) US Power Grid - front view (d) side view

FIG. 5: Power grid networks embedded on the 3D space by using the FR algorithm. Each network is displayed from two distinct viewports: one taken from the two maximum spreading axis, XY, depicted in (**a**) and (**c**); and another from its side, ZY in (**b**) and (**c**). Colors indicate the local dimension of nodes at level $r = 5$. Countries covered by EU Power Grid are shown as background clouds according with the legend of figure 3a.

grid displays very planar structure considering both to the topological and embedding space dimensions, the US power grid is much more less planar topologically. Those results were confirmed by applying a force-directed embedding method. Additionally, local dimension measurements were also seen to reveal borders on such networks.

In summary, the combination of the methodologies exploited in this paper provides new insights about the concept of dimensionality in complex networks. Further investigation shall be done to understand the consequences of the different dimensional structures, with relation to other measurements and dynamics.

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